Two of everything Developing functional thinking in the primary grades through children's literature



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The concept of functional thinking as a foundational idea associated with algebraic thinking is explored by Tracey Muir, Leicha Bragg and Sharyn Livy. They provide ideas for using children's literature as a context to promote functional thinking

Introduction

Traditionally algebra has been regarded as the domain of the secondary school years in Australia and many other countries. Non-mathematics teachers, parents and students often narrowly regard algebra as the manipulation of symbols adhering to tightly prescribed rules (Serow, Callingham & Muir, 2013). It is now recognised, however, that foundational ideas associated with algebraic thinking can, and should be, included in mathematics curricula in the pre-school and primary years (Bobis, Mulligan & Lowrie, 2009). This stance is reflected in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment & Reporting Authority, 2012) which extends key algebraic ideas to patterns and generalisations, and acknowledges that number and algebra are developed together as each enriches the study of the other. This article explores the concept of functional thinking and demonstrates how the story, *Two of Everything* (Hong, 1993) is employed as a springboard for developing functional thinking with students from the early years through to upper primary schooling.

Patterns and algebraic thinking

Mathematics is based on pattern and structure. An understanding of pattern and structure is important in early mathematics learning, with research showing that visualisation skills and structure recognition are positively correlated with mathematical achievement and acquisition of deep conceptual understanding (Mulligan & Mitchelmore, 2009). As described in *Top Drawer Teachers* (Australian Association of Mathematics Teachers, 2013), patterns are important in early years as they lead to many fundamental mathematical ideas, including multiplication, division, equal partitioning and geometric concepts such as symmetry and tessellations (see http://topdrawer.aamt.edu.au/for resources on teaching patterns).

The idea of pattern is central to algebra and understanding the structure of arithmetic forms the basis of algebraic thinking (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2011). Many young children's early experiences of patterning occur through activities that require them to use materials to identify, make, compare and extend repeating patterns (see for example Figure 1). In the early years, the teaching focus is on helping children to identify the part that repeats which provides a foundation for recognising structure and making generalisations. Children's experiences with repeating patterns can be extended to growing patterns, whereby each section experiences consistent growth (Siemon, et al., 2011). While both pattern experiences support algebraic thinking, according to Siemon et al. (2011), repeating patterns lead to multiplicative thinking, whereas growing patterns lead to functional thinking. Growing patterns can be representations of functions if a second variable is introduced which alters the position of each term or shape in the growing pattern. Functional thinking, therefore, focuses on the relationship between two or more varying quantities and is explored more fully in this article.

Figure 1. Foundation student making and recording patterns

Algebra in the Australian Curriculum: Mathematics

The importance of algebra is noticeable in its inclusion as an explicit learning requirement of the *Australian Curriculum* (ACARA, 2012). Number and algebra is one of three prescribed content strands which contains the sub-strand patterns and algebra. The Foundation and Year 1 descriptions both refer to patterning with objects:

> **Foundation:** Sort and classify familiar objects and explain the basis for these classifications. Copy, continue and create patterns with objects and drawings (ACMNA005).

> **Year 1:** Investigate and describe number patterns formed by skip counting and patterns with objects (ACMNA018).

In the later years, the descriptors focus on number patterns, identifying number properties, sequencing and formulation of rules. The teaching of these concepts might be supported through tasks such as making growing patterns for square numbers (see Figure 2). Specifically the Year 5 and 6 descriptions include:

> **Year 5:** Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)

> **Year 6:** Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133).

In addition, algebra is critically related to the development of the *Australian Curriculum Proficiency Strands of Problem Solving and Reasoning* (ACARA, 2012). In particular, when students identify and generalise a pattern, inductive thinking is occurring and conjectures are formulated. For example in Figure 2 a child might predict, “This pattern will continue to grow by an odd number each time, e.g., 3 more, 5 more, 7 more than the last number of blocks.” Conjectures need to be tested, “Was my prediction correct?” If found to apply consistently, reasons need to be established to explain the outcome. Reasoning that explains the origins of a pattern and justify why it must always be true are pervasive to mathematical thinking and lie at the heart of both the reasoning and problem solving proficiency strands of the *Australian Curriculum: Mathematics* (ACARA, 2012).

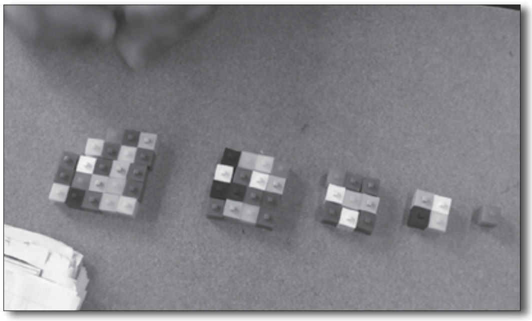

Figure 2. Year 4 students making a growing pattern for square numbers

Functional thinking

Functional thinking focuses on the relationship between two (or more) varying quantities (Siemon, et al., 2011). Siemon et al. (2011) recommended that the learning of functional thinking should begin with young children and focus on the relationships between the operations. In the early years this may involve following rules for consistent changes and reversing this change. For example, a simple rule may be to multiply by 2, as a result the numbers 3, 4, 5 would become 6, 8, 10. Other rules can be added and can differ in complexity: e.g., multiply by 2 and add 1; multiply by itself and take 2. A creative context, such

as those described below, could be used to engage students in function activities.

- **Create a scenario:** Professor Maths has created a number machine. When you put one number in, a different number comes out the other end. One day the Professor put in 4, then 6, then 10. The numbers that came out were 10, 14 and 22. Can you work out what happened to each number before it came out?
- **Input and output box:** Source a large cardboard box that may have contained a fridge or a washing machine with a large hole at the rear of the box for a child to move in and out of comfortably. Cut two slots in the box, one labelled input and one labelled output. One child sits in the box and responds to the cards that are put in the input slot by other children, by generating a 'rule', writing a new number on the back of the card and posting it back in the output slot. The children then work out what the 'rule' is.
- **Guess my rule:** The teacher can prepare a number of cards with numbers on the front and back. As a whole class, show the number on the front of the card and then flip over to show the number on the back. After showing a few examples, the students need to determine the rule, then predict what the next flipped number would be. A variety of different cards can be made, using different colours to signify sets with the same rule; e.g., red cards follow '× 2 + 1'. These could be used as regular warm-up activities.

The above open-ended function activities can be adapted for a range of ages and abilities and could be used to either supplement or extend the experiences described further on in relation to the *Two of Everything* lesson.

Using children's literature as a context

Children's literature can also be a further source for developing functional thinking. Picture books may represent mathematical concepts through their prose, illustrations, logical development and context (Thiessen, 2004) and provide excellent opportunities for engaging in rich mathematical discussions. Very young children can engage with quite sophisticated mathematical concepts, if the context and tasks are appropriate. Kinnear and Clark (2014) found, for example, that 5 year olds were able to engage in probabilistic thinking and make inferences about rubbish data through the use of the story Litterbug Doug. The following provides an account of how the story, *Two of Everything* (Hong, 1993) was employed in an upper primary class to encourage functional thinking. Although the story was used in an upper primary situation, it could readily be adapted for younger children.

Two of Everything

Two of Everything (Hong, 1993) is a Chinese folktale that tells the story of Mr and Mrs Haktak, poor farmers, who unearth a large brass pot with strange powers—it doubles everything placed in it. The Haktaks busy themselves making money by repeatedly placing money in the pot and extracting twice as much. Their seemingly amazing good fortune soon leads to misfortune when Mrs Haktak falls into the pot and Mr Haktak is faced with two wives instead of one. Mr Haktak is equally unsteady on his feet and into the pot he falls. The Haktaks befriend their newest additions to the household and build a life together. The locals have noticed that the Haktaks have become so rich that they can afford two of everything, even themselves!

The story provides a context to consider the magic pot as a function machine which determines the relationship between the input and the output. Wickett, Kharas, and Burns (2002) drew on the opportunity the story offered in developing functions to create a rich task. The following describes the second author's adaption of Wickett et al.'s task to promote an understanding of algebraic thinking with a particular emphasis on functions through the use of children's literature in Years 5 and 6 (students aged 10–12).

Two of Everything lesson

The lesson commenced with reading Two of Everything to students. The students were excited about the possibility of owning such a pot and the potential to accumulate a great deal of money in a short amount of time due to doubling the input each time. A t-chart was drawn on the board to display the function of the pot and show the relationship between the input and output. Next the students were introduced to the teacher's

magic pot—a Halloween witch's pot was used. The students were told the pot was not a doubling pot, but that it did something different. One gold coin was placed inside the pot and out came three gold coins. This information was inserted into a new t-chart. The students were asked to consider what was happening inside the pot. The initial responses were "a tripling pot" or "times by three pot". Further possible responses were sought. The students suggested an "add two pot". The students were asked how they might write the actions of the pot in a shortened form for any number placed in the pot. They suggested IN × 3 and IN + 2 to match the heading of the t-chart. These shortened forms were introduced as the pot's rule.

Next, the class predicted the output of two gold coins placed in the pot when applying the current offered rules. The students were expecting to see either six (IN × 3) or four (IN + 2) gold coins emerge. They were surprised when five gold coins appeared. Neither rule worked. This information, IN = 2 OUT = 5, was added to the t-chart. The students were challenged to consider other possible rules to apply to the teacher's pot. Some children developed a rule that only applied to the relationship between the input of two gold coins and output of five gold coins and neglected to consider the relationship between the input of one gold coin and output of three gold coins. The teacher referred these students back to the pot in the Two of Everything story whose doubling rule was applied consistently to any objects placed in the pot. The key teaching point was one rule applied to all inputs.

A new rule suggested for the pot was IN + IN + 1. Another child suggested that the rule could be simplified to IN × 2 + 1. At this stage, the teacher decided not to simplify the rule further but rather to consolidate the students' new algebraic thinking. This rule was checked if it could be applied successfully to the existing input and output on the t-chart and it was. The teacher introduced the idea of replacing the IN with a shorter variable. One child revealed his dad had explained you can use a letter to represent any number. Thus the new rule was written as n × 2 + 1. The suggestion of employing a letter as a variable might not arise in the class, therefore, a teacher might decide to replace the IN with a shape, such as a triangle, or suggest a single letter themselves. However, to date, on every occasion this lesson has been conducted, one student suggests using a letter. Next, a prediction and testing of the rule n × 2 + 1 for the input of three was undertaken and the students were satisfied that they had uncovered the mystery rule of the teacher's pot when seven gold coins appeared. One more test took place for an input of four, and nine gold coins appeared. Eureka! These results were recorded progressively on the t-chart and the students were asked to determine how many gold coins would appear if 5, 10, 12, 20, 100 gold coins were placed in the pot.

In the next stage of the lesson the students were invited to create a rule for their own magic pot. In pairs they were to create the rule, write it on one side of a piece of paper and draw a t-chart on the other side providing at least four different examples of the input and the effect on the output as a result of applying their rule. The open-ended nature of this task provided students with the opportunity to create rules that they were at ease solving. Figure 3 is an example of a simple one-step rule. The rule could be written using the variable of IN or a letter notation. The students determined the variable they were most comfortable with applying.

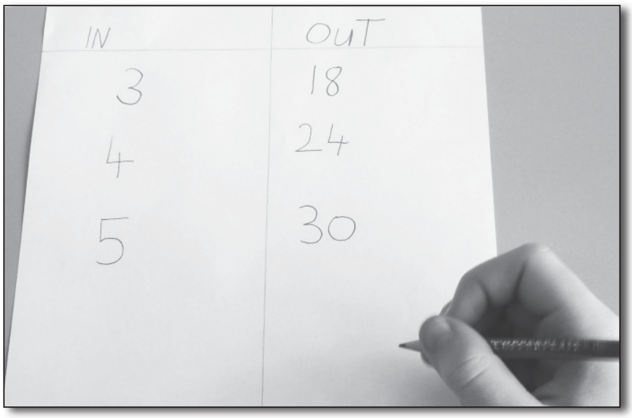

Figure 3. T-chart for the one step rule n × 6.

Once the rule and t-chart were completed the pairs would trade their t-chart with another pair of students and attempt to unlock the others' mystery rule. Great excitement ensued as the pairs tried to design rules that were challenging for others. As illustrated in Figure 4 some students unexpectedly did not provide sequential numbers for their input; e.g., 1, 2, 3, 4 but rather 5, 9, 14. This random approach to the selection of the input numbers was beneficial in promoting an understanding of functions as the relationship

between the input and output through scanning horizontally across the t-chart, rather than the students focusing on the vertical numbers to recognise a pattern and predict the next number in the sequence.

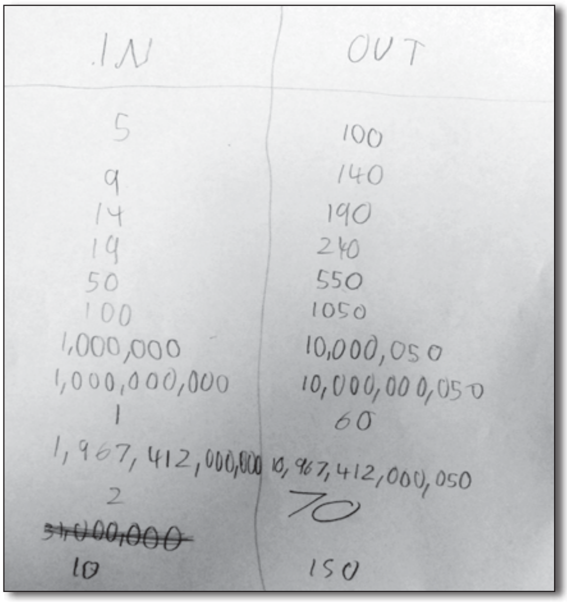

Figure 4. T-chart for the rule n × 10 + 50.

The room was buzzing with children running from one pair to the next to see if they could discover their rule. Many of the rules were challenging, however, some children created rather imaginative and detailed rules which were unfortunately too difficult to crack. For example using numbers in the billions and trillions, or including all four operations in the rule. While typically a teacher does not want to restrict students' creativity, at times it is necessary, especially when first trialling this task, so that students can access the task in a meaningful way. Therefore, in some instances, restrictions that made the task richer and attainable were enacted, such as, "Select numbers from 1 to 5 and use a maximum of two steps in your rule". While initially disappointed the students were soon pleased to witness their classmates more engaged in their magic pot, rather than taking one look at their t-chart, determining it was too difficult to consider and walking away to the next pair's pot.

Some rules opened up a dialogue about the inverse relationship of addition and subtraction. For example, one pair of students created a rule that entailed subtracting then adding numbers; e.g., IN × 12 – 13 + 32. The students witnessed in their own instances how adding and subtracting numbers one step after the other can be reduced to a single step. On occasion students were not willing to alter their inventive rule but accepted their peer's simplified answer; e.g., IN × 12 + 19.

In the course of the lesson the teacher assessed the complexity of the students' thinking through inviting them to share the strategies they employed to uncover the rules of different magic pots. Naïve to complex thinking was noted and at the conclusion of the lesson, the teacher prompted these students to share their experience of the task and their strategies, building upon the complexity of the strategies in a progressive manner. The most common strategy was to initially scan the input and output numbers on the t-chart for any patterns that connected these numbers to their experience of the multiplication tables. While these children searched across the t-chart to find a pattern, others focused on the relationship between a single input and output value, considered a rule that would apply to these two values and moved to the next pair of input and output values to utilise the rule in that situation.

Adapting and extending the task

A follow-up to this task would be to pose questions to challenge students' thinking, such as, "How many inputs would it take for you to reach 50 [180 or one million or another suitable number] gold coins if you started with five gold coins in the Haktak's doubling pot?" "Is it possible to reach exactly 100 gold coins using the Haktak's doubling pot, if you started with one gold coin? Five gold coins? Seven gold coins? If so, show how you know for sure." To increase the challenge, the Haktak's pot in these questions would be replaced with the students' own magic pot. The task could also be easily adapted for younger students—beginning with the doubling idea and then extending to different one step inputs and outputs as appropriate. Suitable manipulatives such as unifix blocks, counters, and straws, may be employed to support children's learning.

Conclusions

The development of functional thinking begins in the early years and can be extended through the creation of meaningful contexts, such as those provided through children's literature. The lesson account shows how students can readily engage

with a good story and are motivated to explore mathematical content and proficiency skills such as problem solving and reasoning. The domain of algebra is not one that should be relegated to the secondary school syllabus, but meaningfully developed from the early years onwards.

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